

CS103, AUTUMN 2026



Lecture 06: **Functions**

Part 1 of 2

Functions

Part 1

1. Announcements
2. Functions: Intuitions and Examples
3. Domains and Codomains
4. Official Rules for Functions
5. Ways to Define Functions
6. Involutions (with Proofs!)
7. The Assume-Prove Table
8. Injections, Surjections, and Bijections
9. What's Next?

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Things to Have On Your Radar

- ***Regret Clause Forms***
 - due Tuesdays at 1:00 PM
- ***Recitation Swaps and Final Opt-Out***
 - due this coming Sunday, 11:59 PM, via the snazzy new **Recitation Dashboard**
- ***Left-Handed Exam Desk Form***
 - due Monday of next week
- ***Lecture Attendance (PolEV) Opt-Out***
 - Opens sometime next Monday (Oct. 12), due next week on Friday (Oct. 16) at 11:59 PM.

Other Recitationy Things

- ***Check Your Recitation Status***

- Double check that your enrollment / opt-out status is as expected on the **Recitation Dashboard**.

- ***New Recitation Page on Course Website***

- This is where we'll post recitation worksheets and solutions moving forward.
- Check out the FAQ we posted there for additional information / policies.

- ***Tuesday Morning Recitation Locations***

- Tuesday 10:30 AM has moved to Gates 100
- Tuesday 11:30 AM has moved to ??? (will post soon)



These are
stuffed

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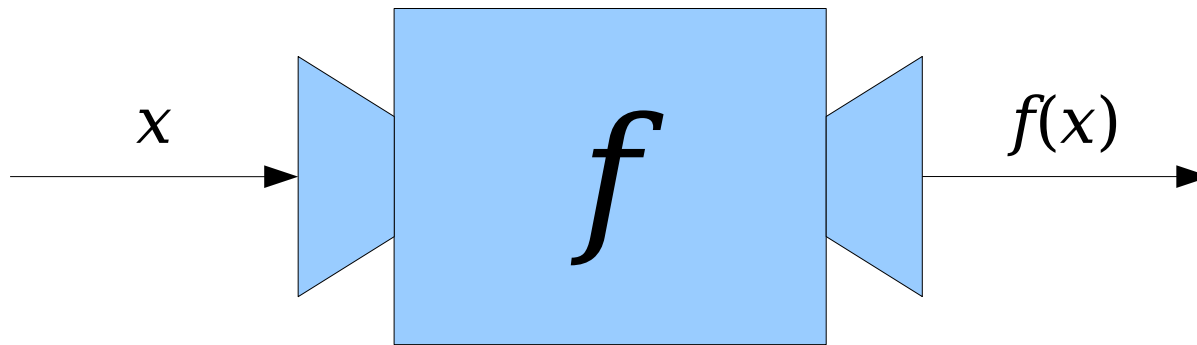
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What is a function?

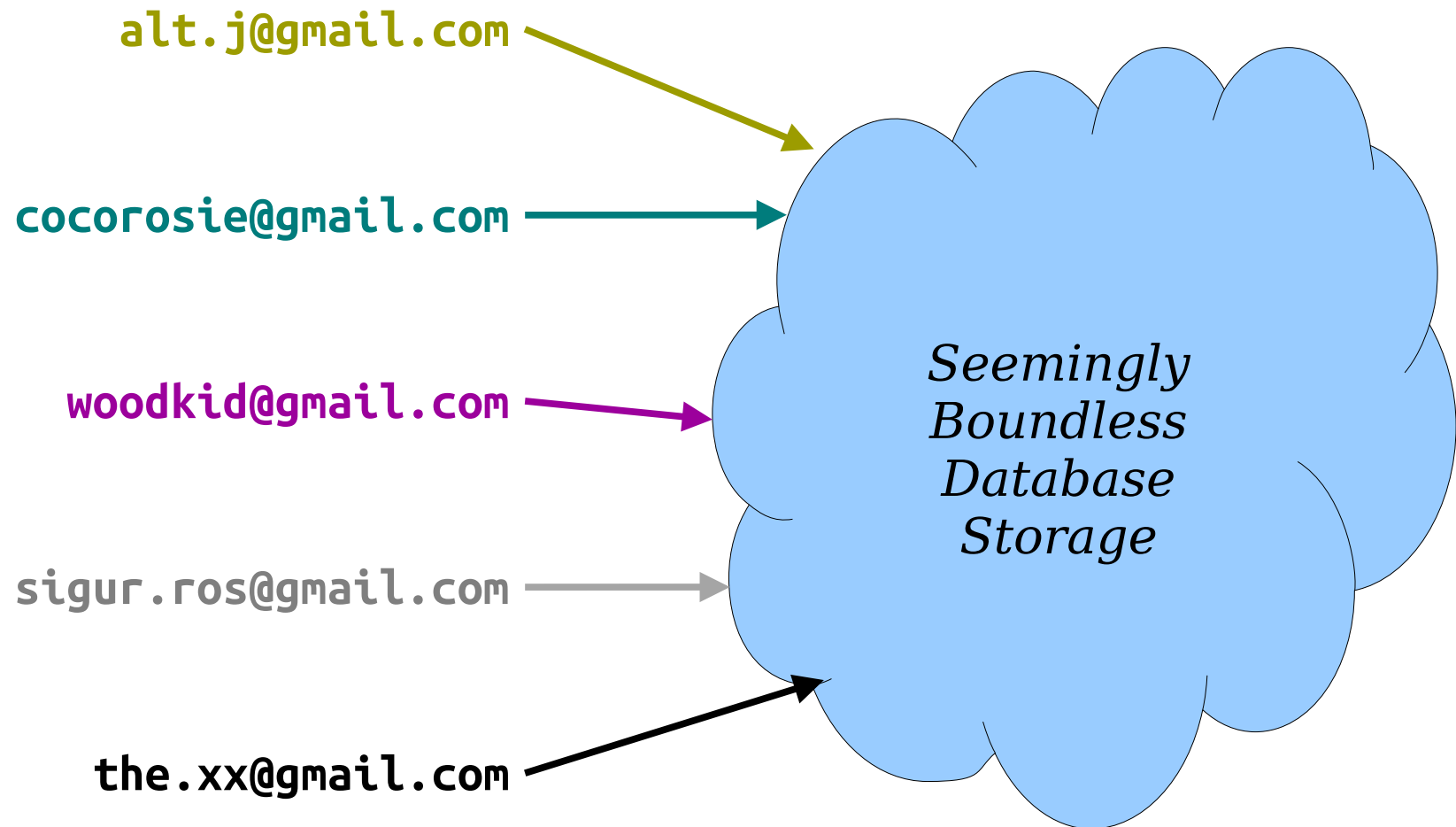
High-Level Intuition:

A function is an object f that takes in exactly one input x and produces exactly one output $f(x)$.

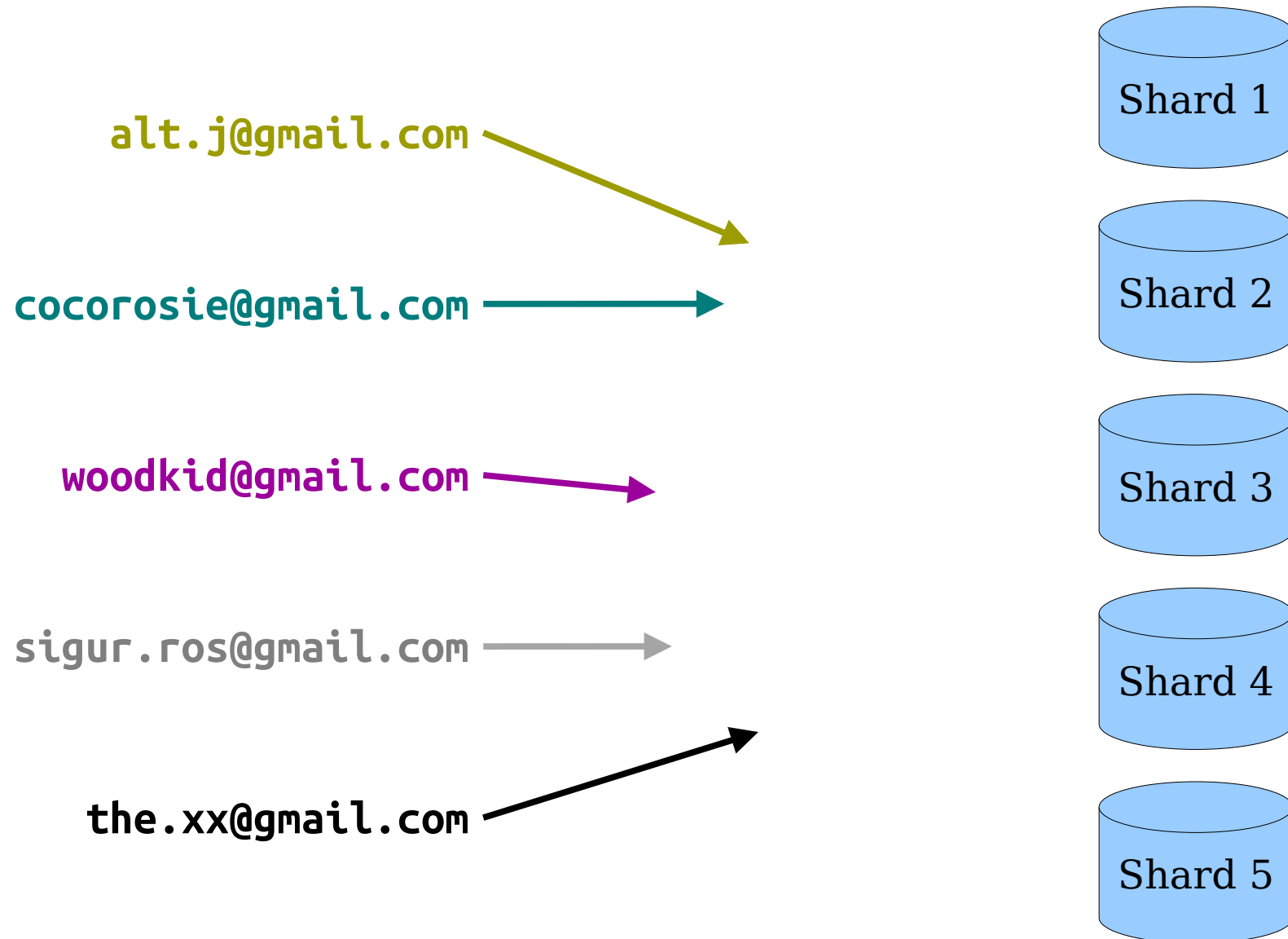


(This is not definition. It's just to help you build and intuition.)

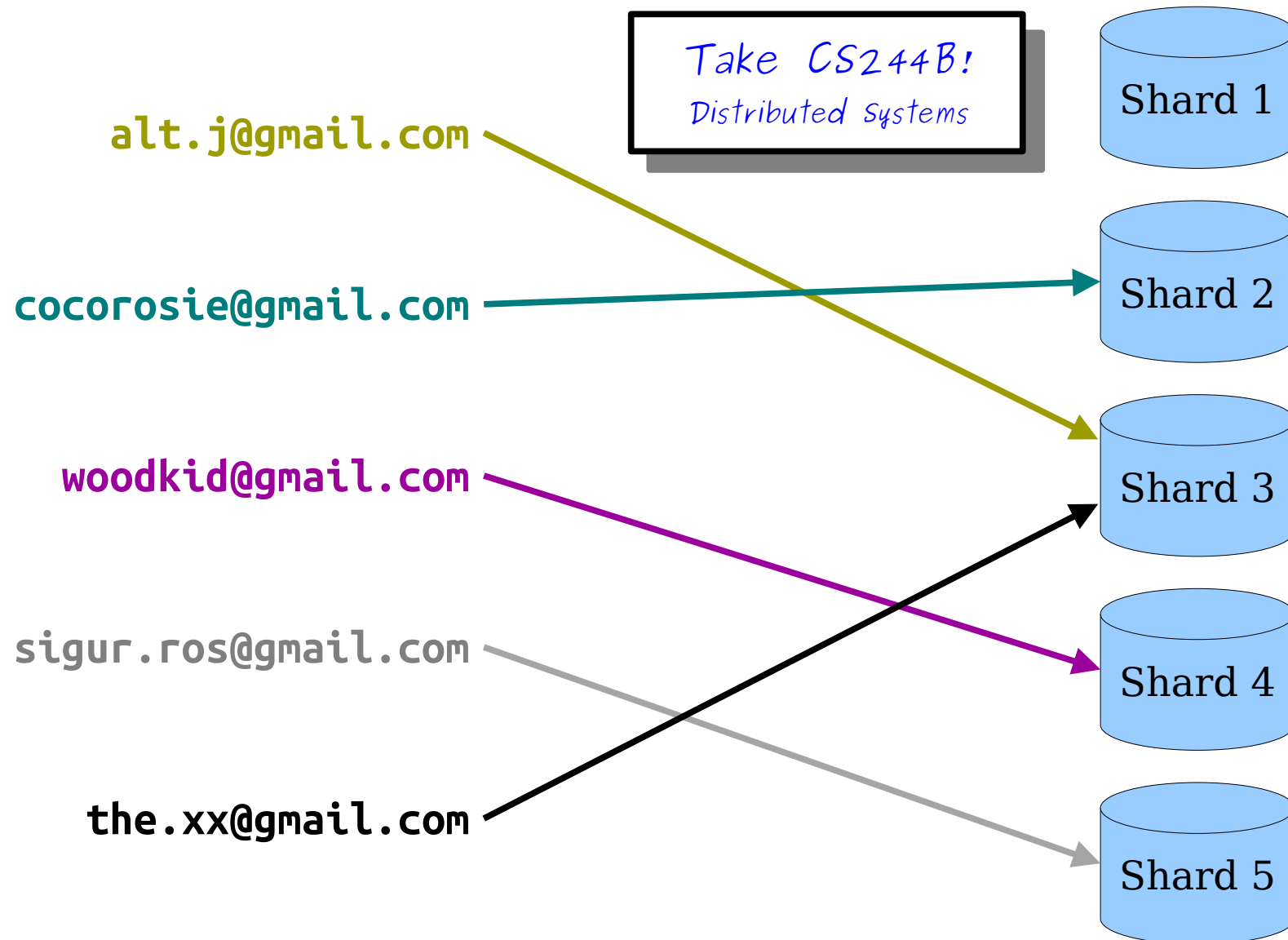
Example: Database Sharding



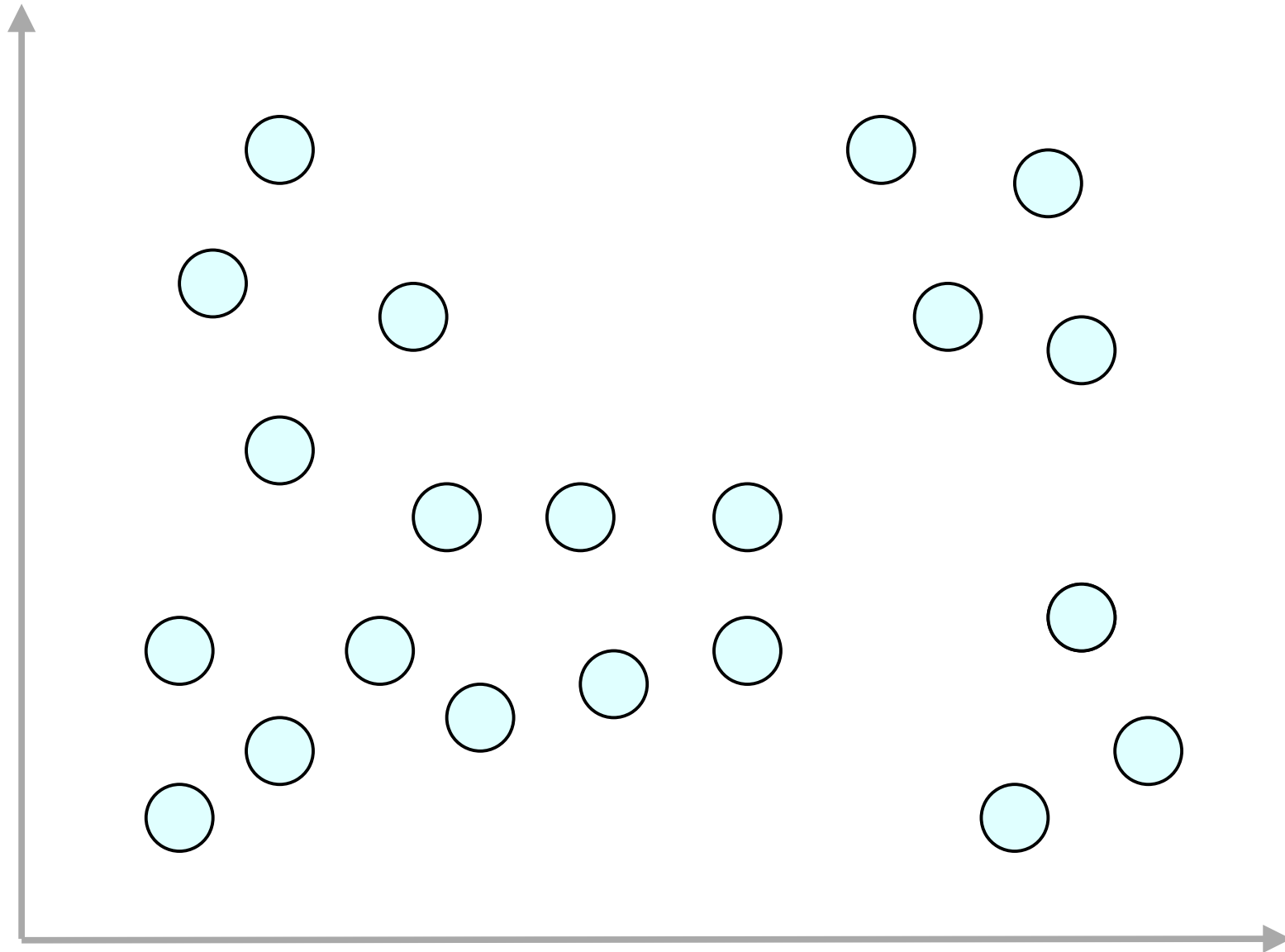
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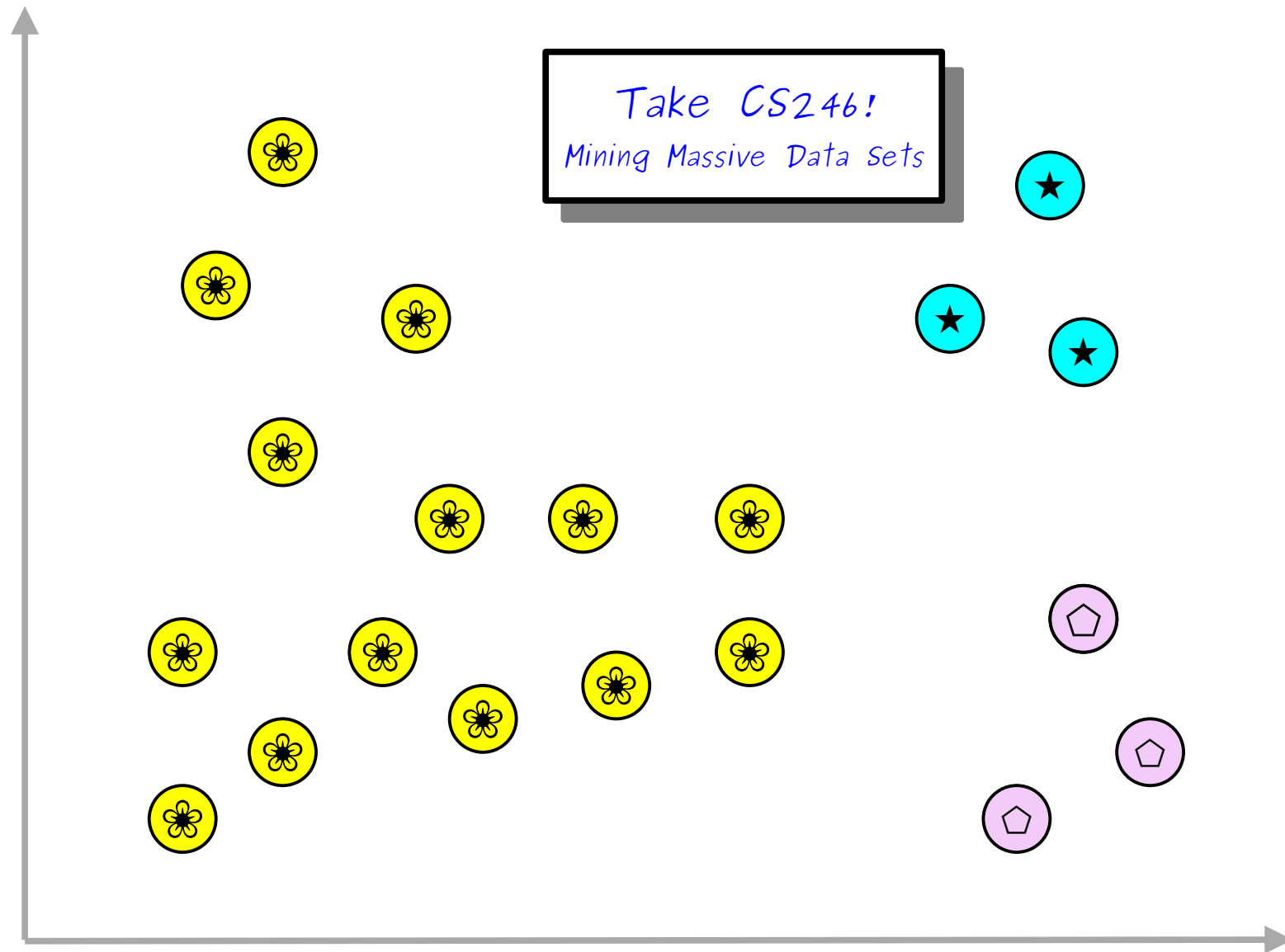
Example: Database Sharding



Example: Data Clustering



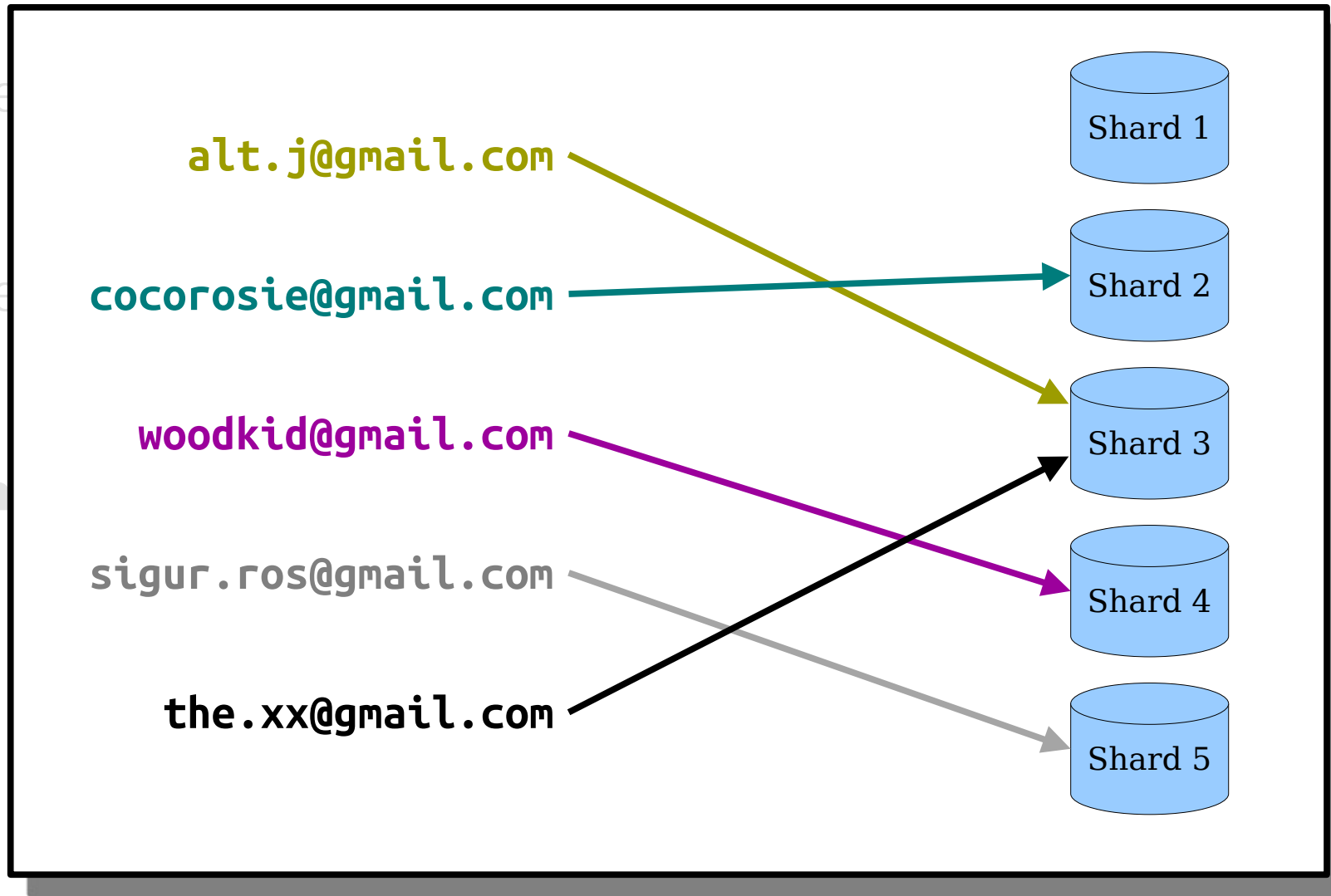
Example: Data Clustering



What's In Common?

- We have a **fixed, known set of possible inputs**.
 - In our examples: user names and 2D data points.
- We have a **fixed, known set of possible outputs**.
 - In our examples: database shards and cluster labels.
- **Each input** is assigned an output.
 - Some outputs might be assigned multiple inputs.
 - Some outputs might be assigned no inputs.

What's In Common?



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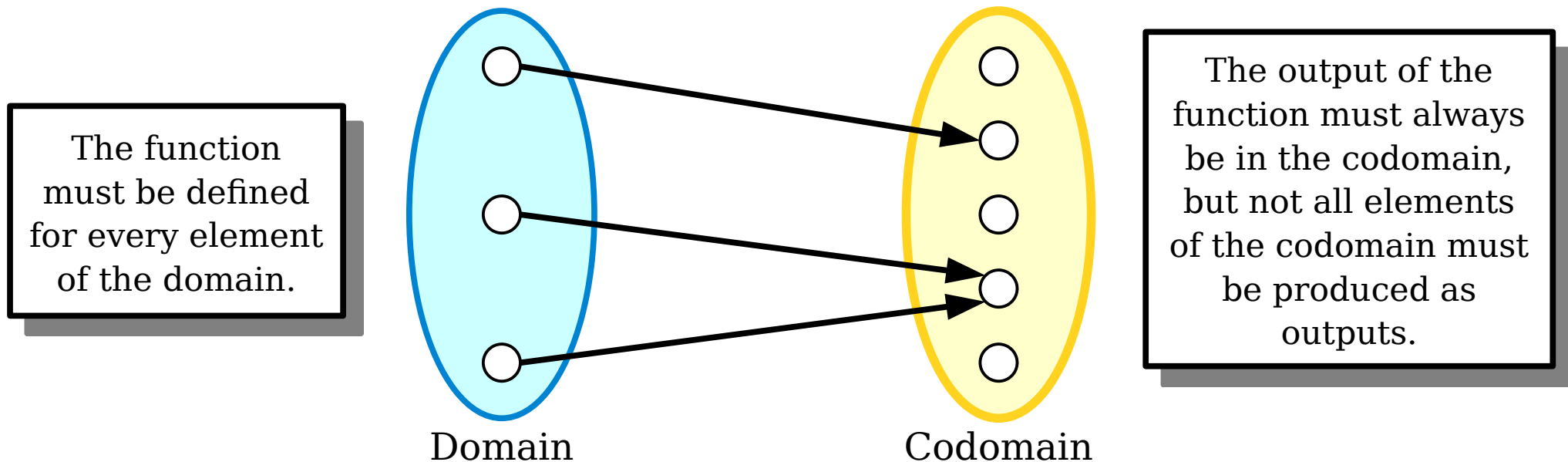
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Domains and Codomains

- Every function f has two sets associated with it: its **domain** and its **codomain**.
- A function f can only be applied to elements of its domain. For any x in the domain, $f(x)$ belongs to the codomain.



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The **domain** of this function is $\mathbb{R}$. Any real number can be provided as input.

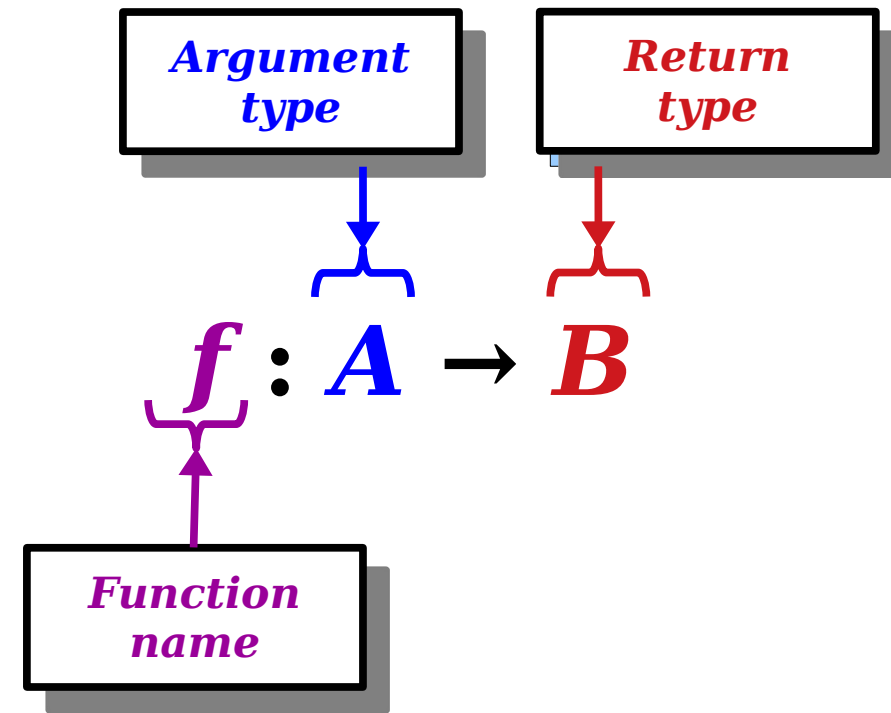
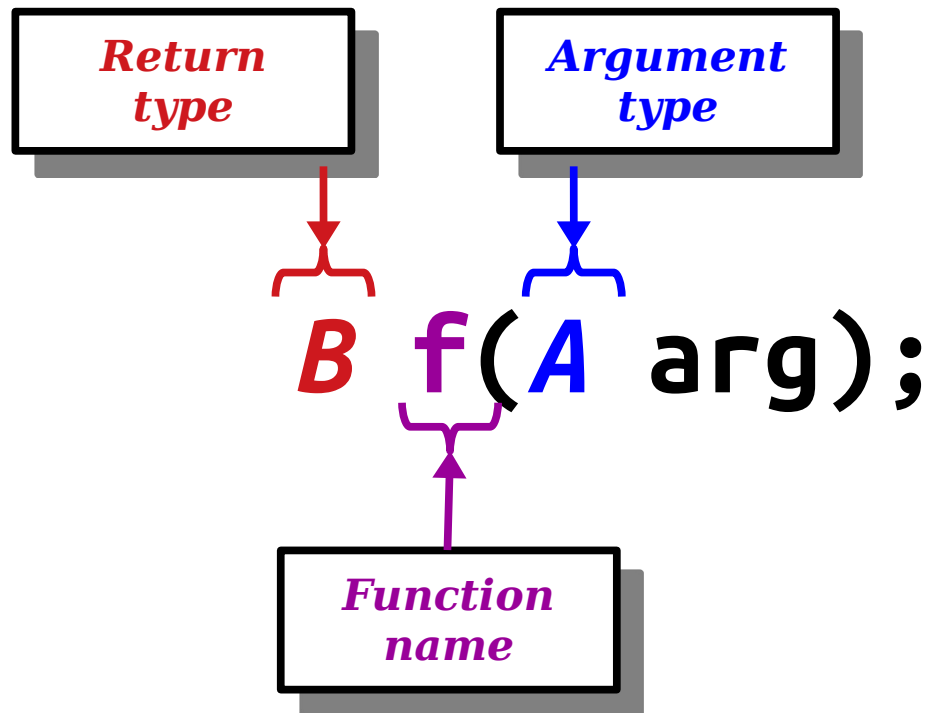
The **codomain** of this function is $\mathbb{R}$. Everything produced is a real number, but not all real numbers can be produced.



```
double absoluteValueOf(double x) {  
    if (x >= 0) {  
        return x;  
    } else {  
        return -x;  
    }  
}
```

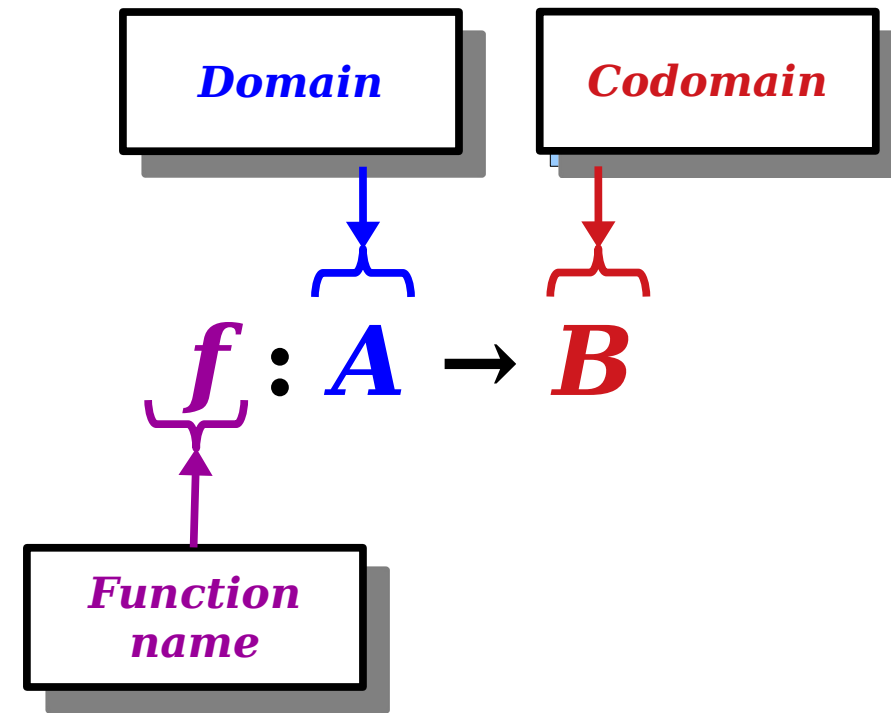
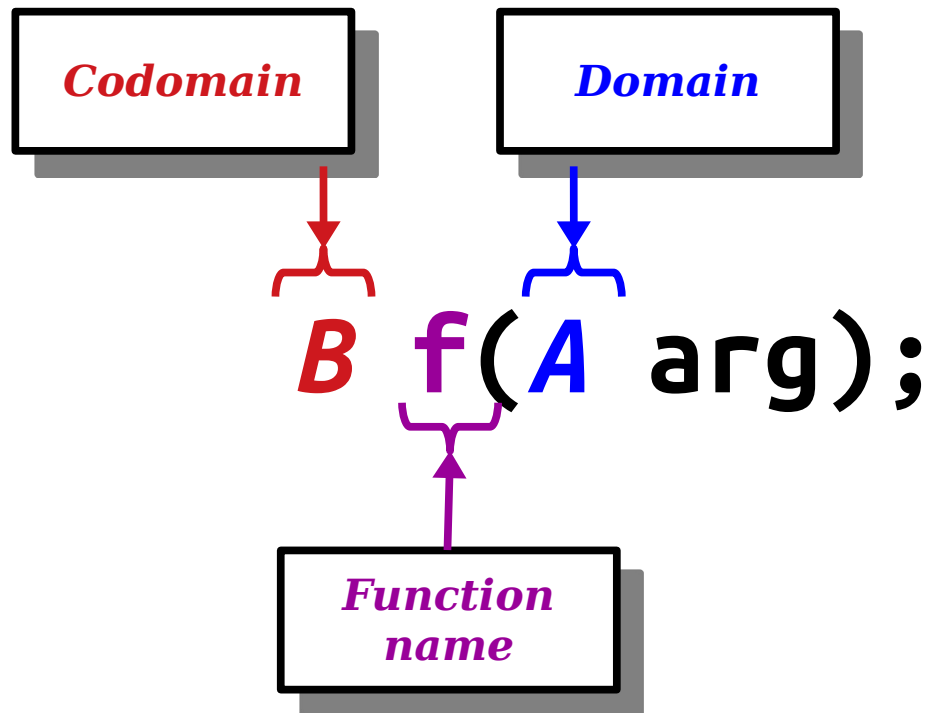
Domains and Codomains

- If f is a function whose domain is A and whose codomain is B , we write $f : A \rightarrow B$.
- Think of this like a “function prototype” in C++.



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- Think of this like a “function prototype” in C++.



Some Observations

- Usually, when working with functions, you pick the domain and codomain before defining the rule for the function.
 - Think programming: you usually know what types of things you're working with before you know how they work.
- In mathematics, all functions take in exactly one argument: an element of the domain.
 - If you're clever, you can get two or more arguments to a function while still obeying this rule. Chat with me after class to learn more!
- In mathematics, functions are **deterministic** and can't behave randomly.
 - If you're clever, you can get functions that kinda sorta ish look random. Chat with me after class to learn more!

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The Official Rules for Functions

- Formally speaking, we say that $f : A \rightarrow B$ if the following two rules hold.
- First, f must obey its domain/codomain rules:

$$\forall a \in A. \exists b \in B. f(a) = b$$

(“Every input in A maps to some output in B .”)

- Second, f must be deterministic:

$$\forall a_1 \in A. \forall a_2 \in A. (a_1 = a_2 \rightarrow f(a_1) = f(a_2))$$

(“Equal inputs produce equal outputs.”)

- If you’re ever curious about whether something is a function, look back at these rules and check! For example:
 - Can a function have an empty domain?
 - Can a function have an empty codomain?

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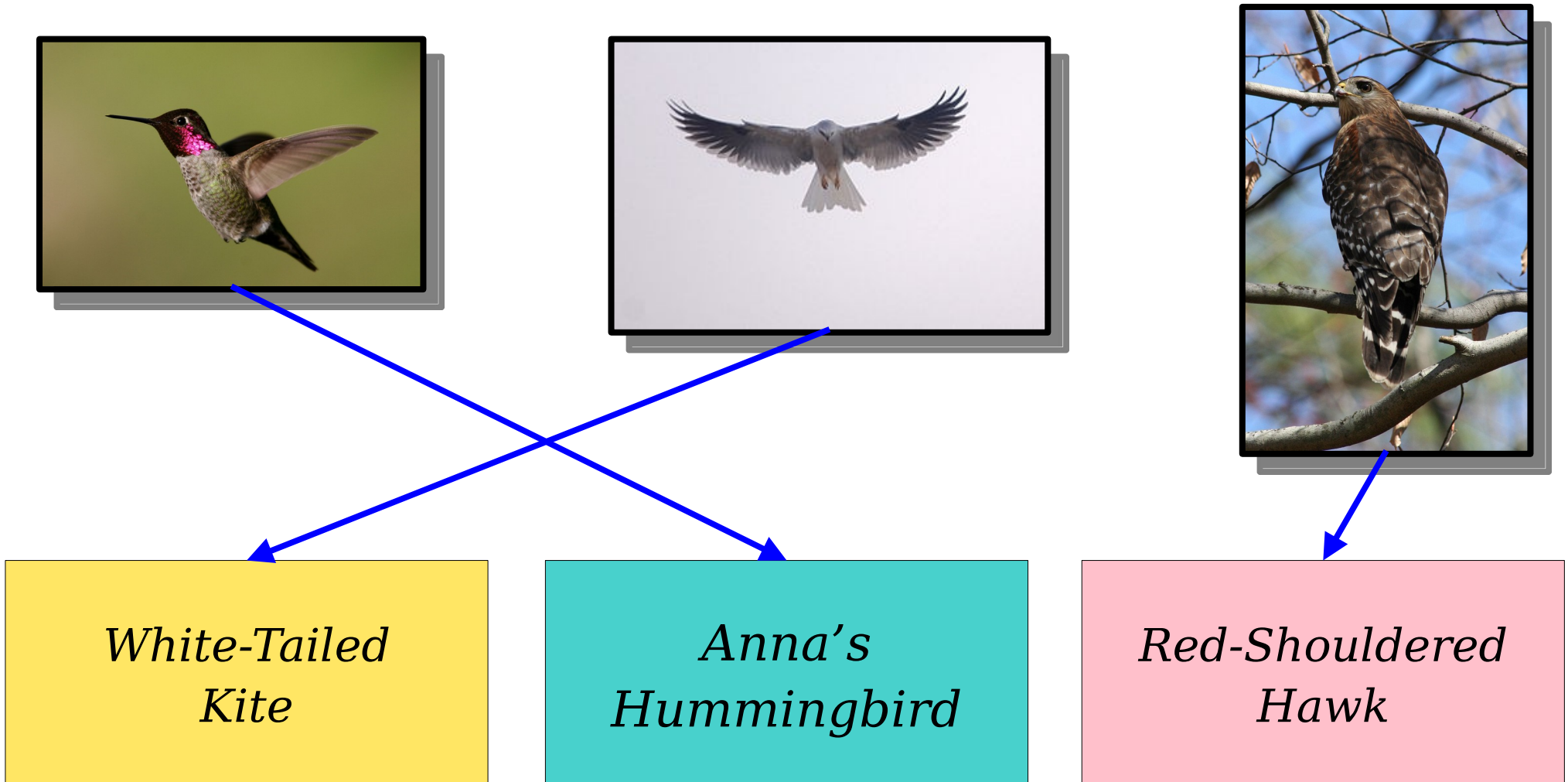
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Defining Functions

- To define a function, you need to
 - specify the domain,
 - specify the codomain, and
 - give a **rule** used to evaluate the function.
- All three pieces are necessary.
 - We need to domain to know what the function can be applied to.
 - We need to codomain to know what the output space is.
 - We need the rule to be able to evaluate the function.
- There are many ways to do this. Let's go over a few examples.

Defining Functions



Functions can be defined as a **picture**. Draw the domain and codomain explicitly. Then, add arrows to show the outputs.

Defining Functions

$f : \mathbb{Z} \rightarrow \mathbb{Z}$, where

$$f(x) = x^2 + 3x - 15$$

Functions can be defined as a **rule**.
Be sure to explicitly state what the
domain and codomain are!

Defining Functions

$f : \mathbb{Z} \rightarrow \mathbb{N}$, where

$$f(n) = \begin{cases} n & \text{if } n \geq 0 \\ -n & \text{if } n \leq 0 \end{cases}$$

Some rules are given *piecewise*. We select which rule to apply based on the conditions on the right. (Just make sure at least one condition applies and that all applicable conditions give the same result!)

Some Nuances

$f : \mathbb{R} \rightarrow \mathbb{R}$, where

$$f(x) = \frac{x+2}{x+1}$$

Answer at

cs103.stanford.edu/pollev

Is this a function from $\mathbb{R}$ to $\mathbb{R}$?

Some Nuances

$f : \mathbb{R} \rightarrow \mathbb{R}$, where

$$f(x) = \frac{x+2}{x+1}$$

Answer at

cs103.stanford.edu/polleev

This expression isn't defined when $x = -1$, so f isn't defined over its full domain. We therefore don't consider it to be a function.

Is this a function from $\mathbb{R}$ to $\mathbb{R}$?

Some Nuances

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Some Nuances

$f : \mathbb{N} \rightarrow \mathbb{R}$, where

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Answer at

cs103.stanford.edu/pollev

Yep, it's a function! Every natural number maps to some real number.

Is this a function from $\mathbb{N}$ to $\mathbb{R}$?

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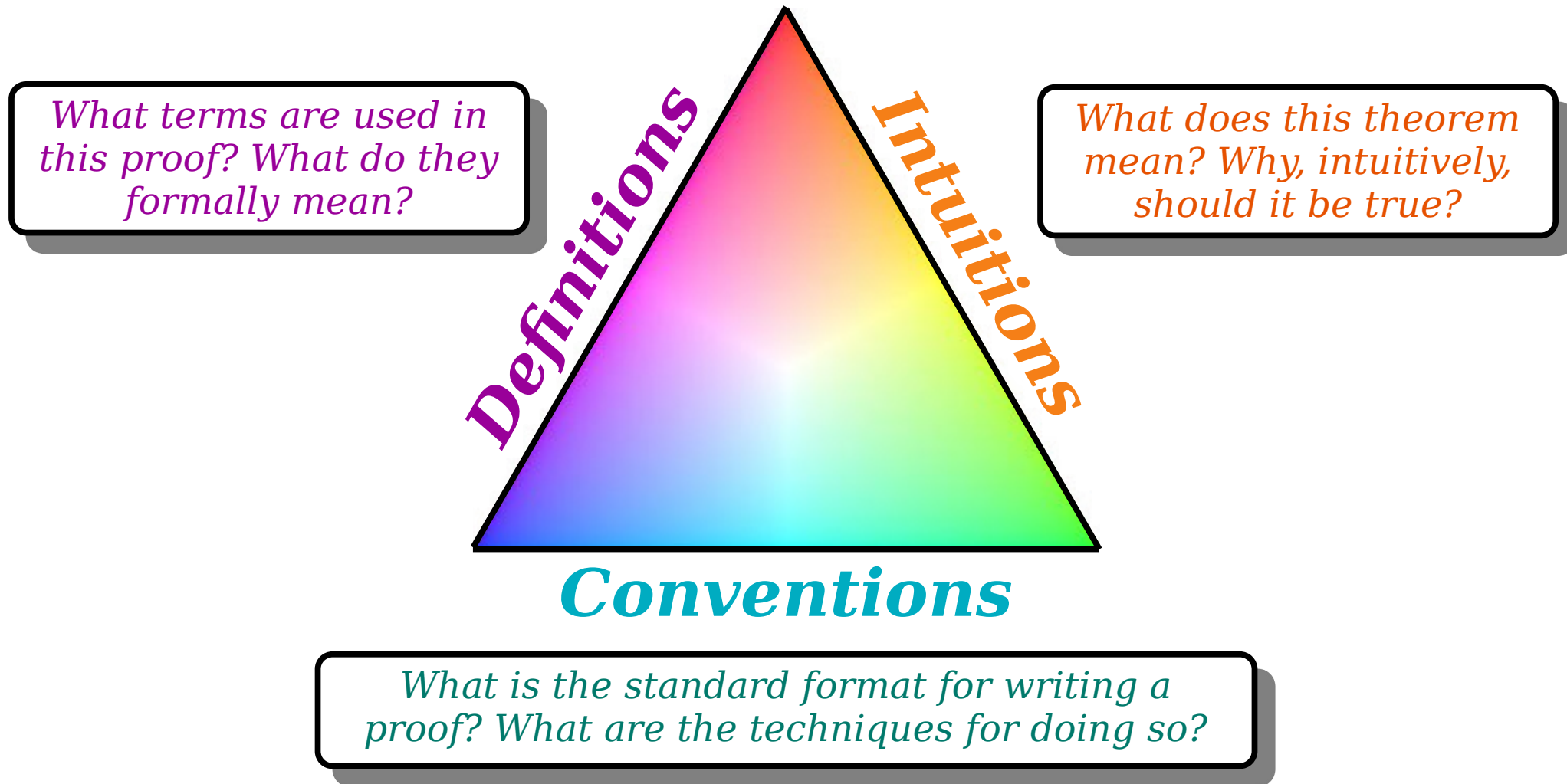
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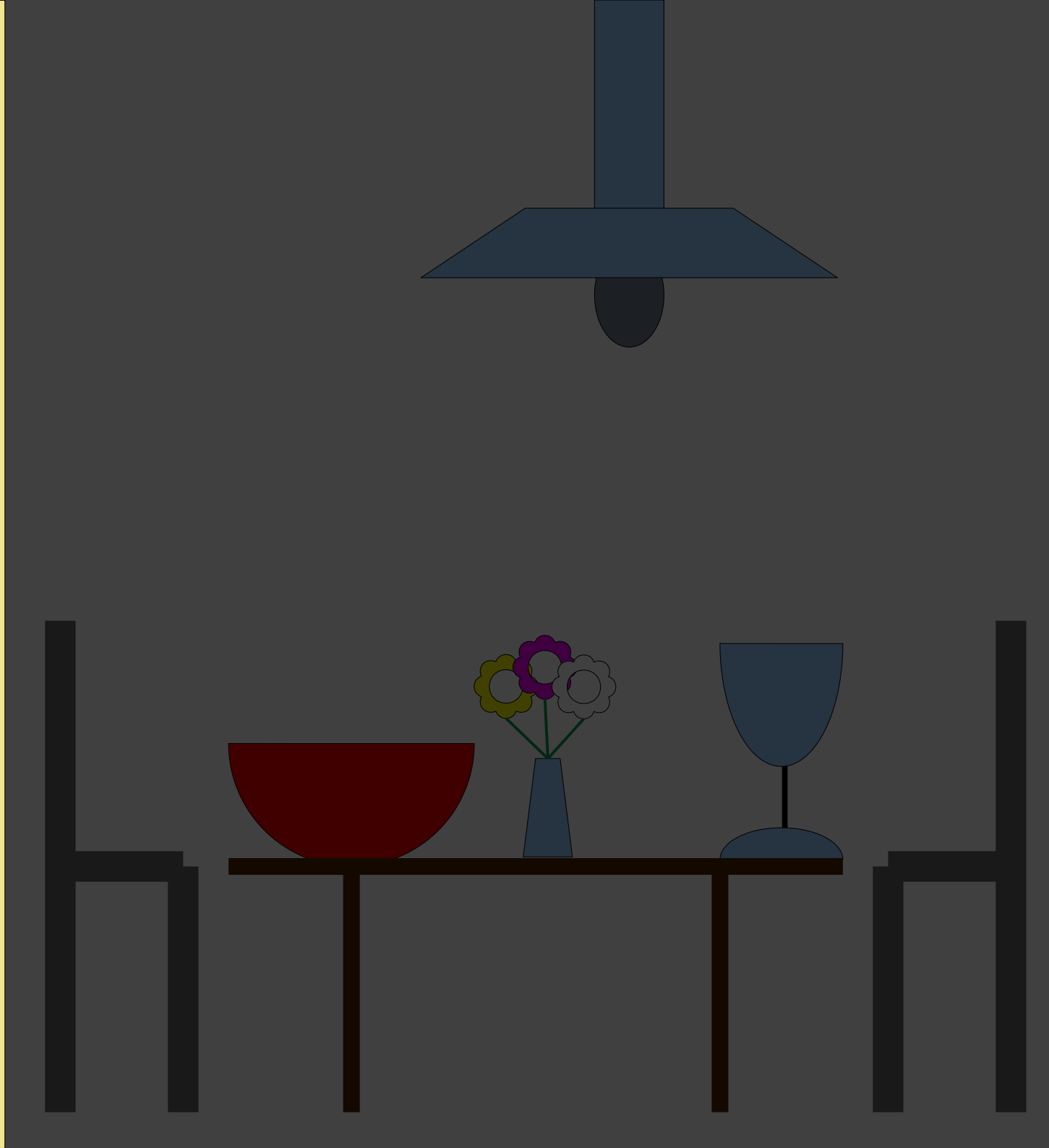
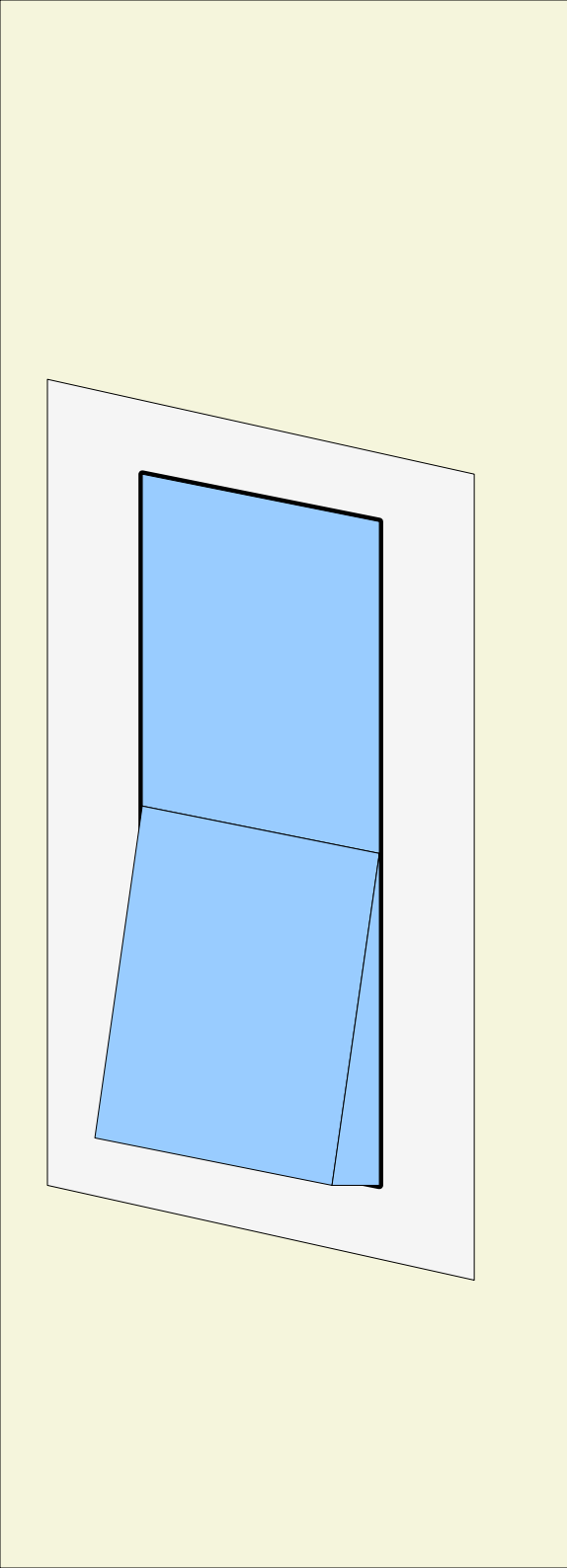
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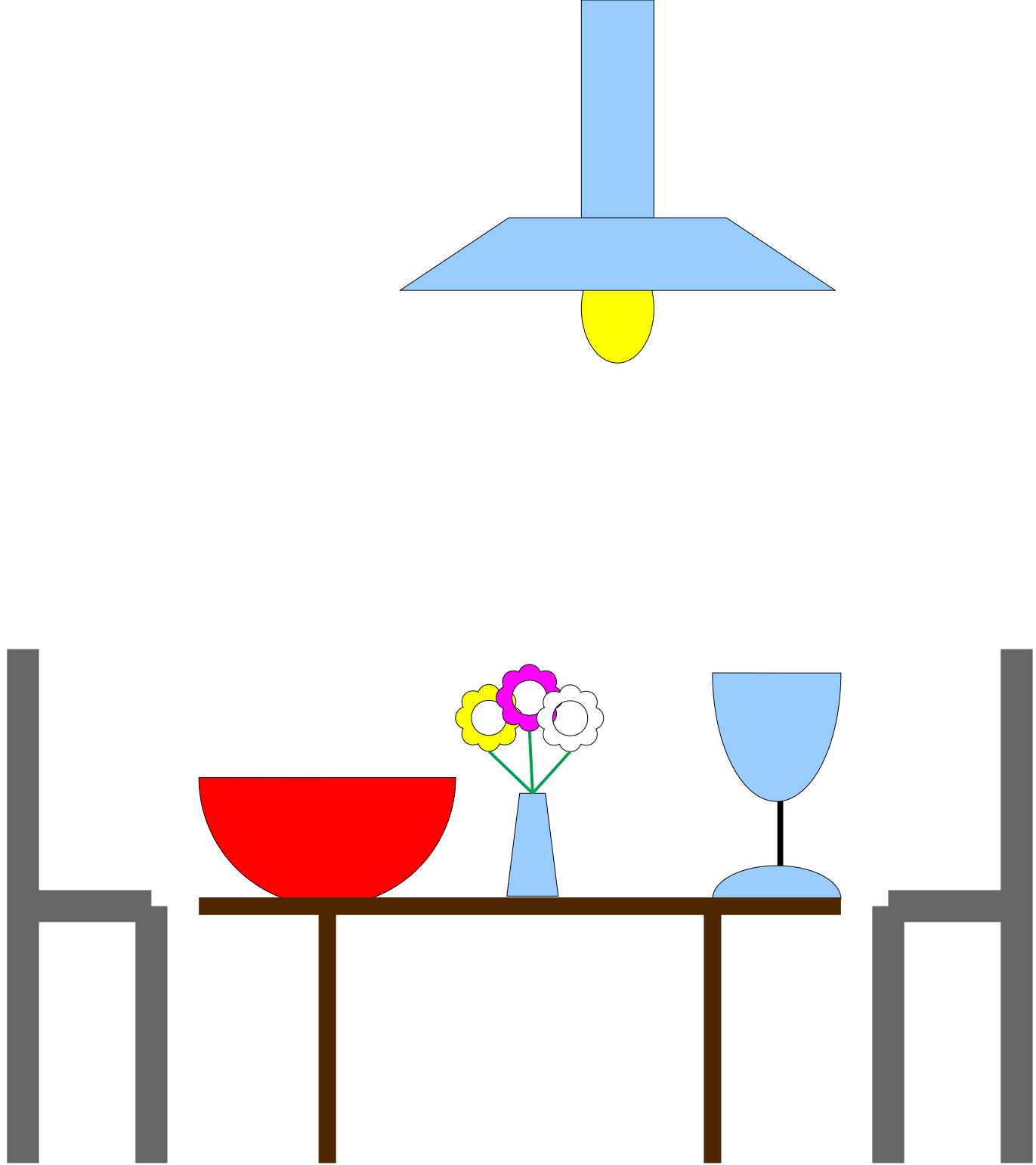
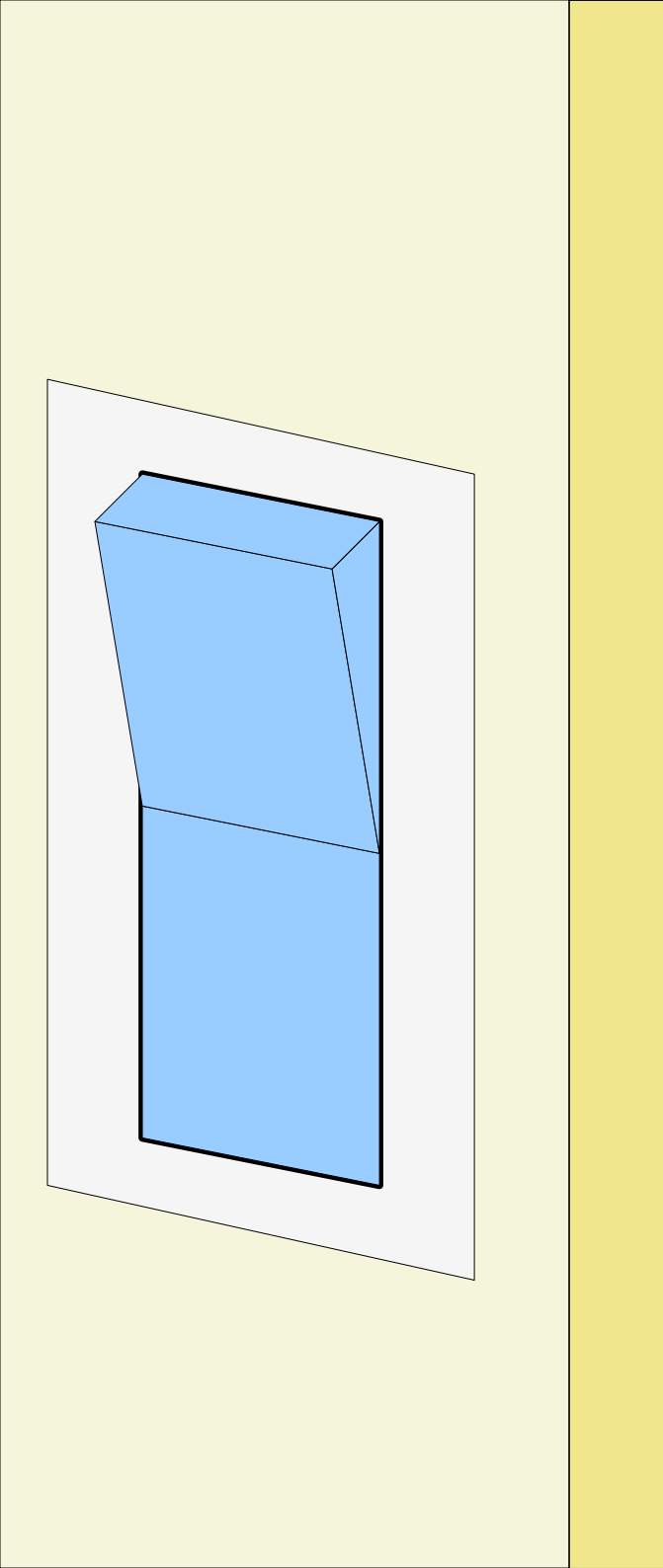
The Proof Writing Triad



Undoing by Doing Again

- Some operations invert themselves. For example:
 - In first-order logic, $\neg\neg A$ is equivalent to A .
 - In algebra, $-(-x) = x$.
 - Flipping a switch twice is the same as not flipping it at all.



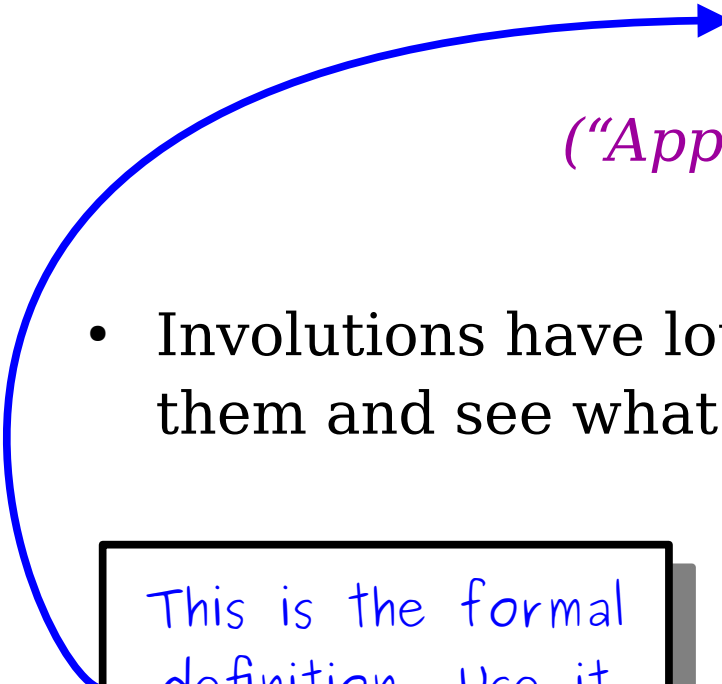


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 - In algebra, $-(-x) = x$.
 - Flipping a switch twice is the same as not flipping it at all.
 - In set theory, $(A \Delta B) \Delta B = A$. (*Yes, really!*)
- Operations with these properties are surprisingly useful in CS theory and come up in a bunch of contexts.
 - Storing compressed approximations of sets (XOR filters).
 - Building encryption systems (symmetric block ciphers).
 - Transmitting a large file to multiple receivers (fountain codes).

Involutions

- A function $f : A \rightarrow A$ from a set back to itself is called an **involution** when the following first-order logic statement is true about f :


$$\forall x \in A. f(f(x)) = x.$$

(“Applying f twice is equivalent to not applying f at all.”)



- Involutions have lots of interesting properties. Let's explore them and see what we can find.

This is the formal definition. Use it in proofs.

This is just an intuition. Don't use it in proofs.

Involutions

- Which of the following are involutions?
 - $f : \mathbb{Z} \rightarrow \mathbb{Z}$ defined as $f(x) = x$.
 - $g : \mathbb{Z} \rightarrow \mathbb{Z}$ defined as $g(x) = -x$.
 - $h : \mathbb{R} \rightarrow \mathbb{R}$ defined as $h(x) = 1/x$.
 - $p : \mathbb{N} \rightarrow \mathbb{N}$ defined as follows:

$$p(n) = \begin{cases} n+1 & \text{if } n \text{ is even} \\ n-1 & \text{if } n \text{ is odd} \end{cases}$$

Answer at

cs103.stanford.edu/pollerv

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Involutions

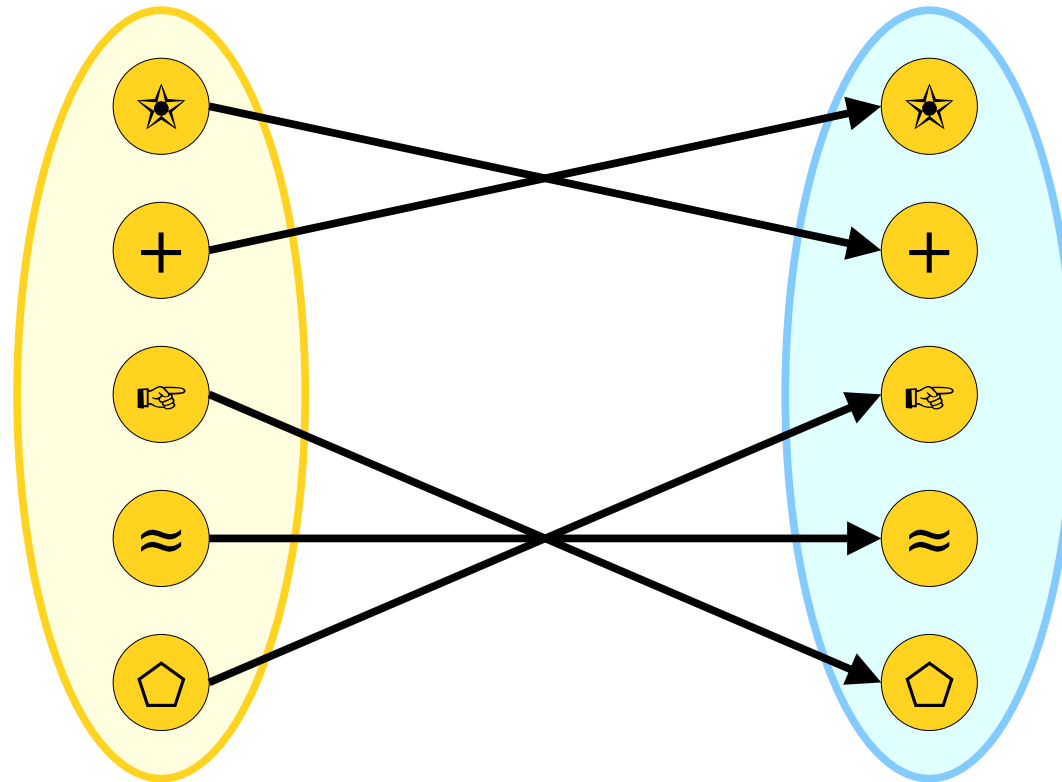
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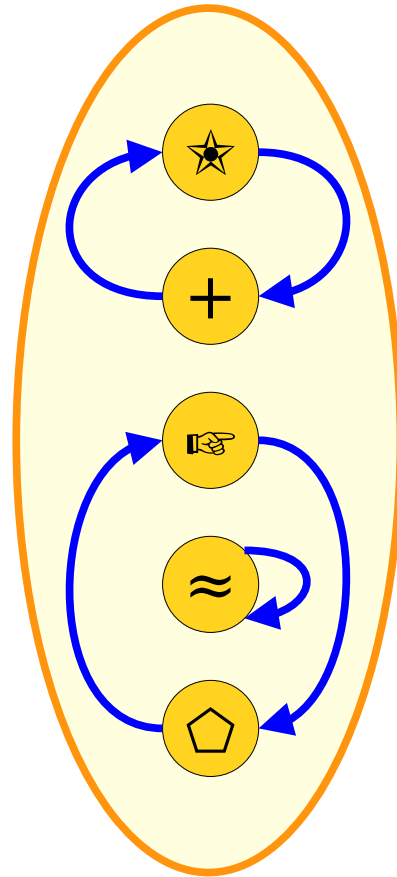
Involutions, Visually



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Involutions, Visually



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Proofs on Involutions

Theorem: The function $f : \mathbb{Z} \rightarrow \mathbb{Z}$ defined as

$$f(n) = \begin{cases} n+1 & \text{if } n \text{ is even} \\ n-1 & \text{if } n \text{ is odd} \end{cases}$$

is an involution.

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This proof contains no first-order logic syntax (quantifiers, connectives, etc.). It's written in plain English, just as usual.

Functions

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4. Official Rules for Functions
5. Ways to Define Functions
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**A REALLY
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The Assume-Prove Table

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[illegible]

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What does it mean for f to be an involution?

$$\forall n \in \mathbb{N}. f(f(n)) = n.$$

What is the negation of this statement?

$$\begin{aligned} &\neg \forall n \in \mathbb{N}. f(f(n)) = n \\ &\exists n \in \mathbb{N}. \neg (f(f(n)) = n) \\ &\exists n \in \mathbb{N}. f(f(n)) \neq n \end{aligned}$$

Therefore, we need to find some concrete choice of n such that $f(f(n)) \neq n$.

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Proof: We need to show that there is some $n \in \mathbb{N}$ where $f(f(n)) \neq n$.

Pick $n = 2$. Then

$$\begin{aligned} f(f(n)) &= f(f(2)) \\ &= f(4) \\ &= 16, \end{aligned}$$

which means that $f(f(n)) \neq n$, as required. ■

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Functions

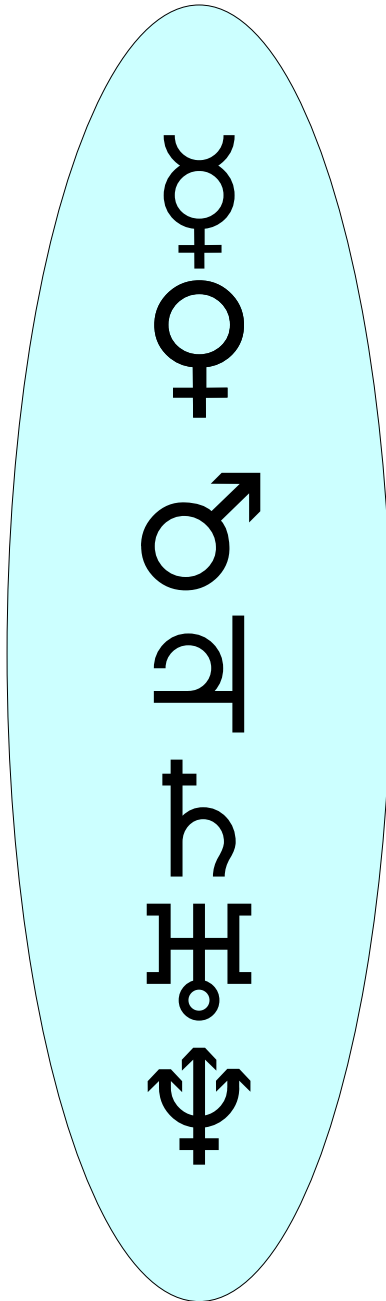
Part 1

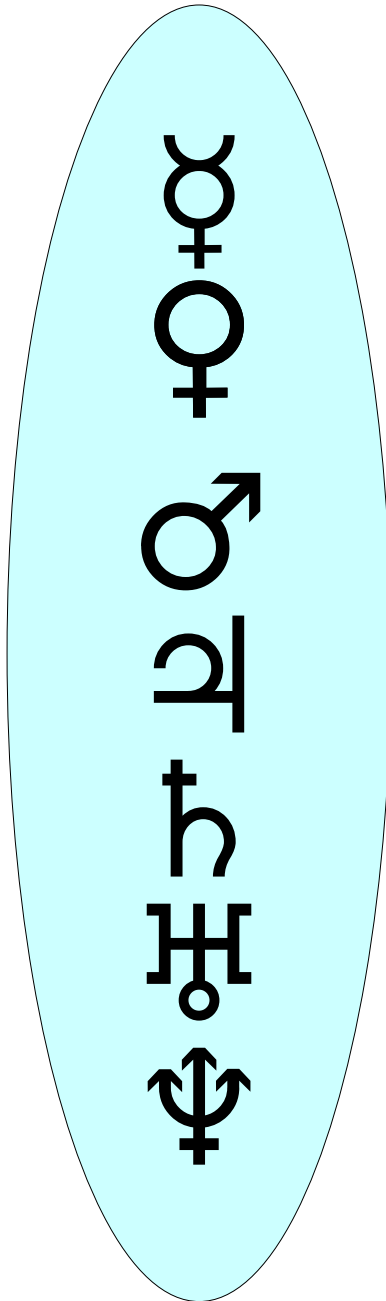
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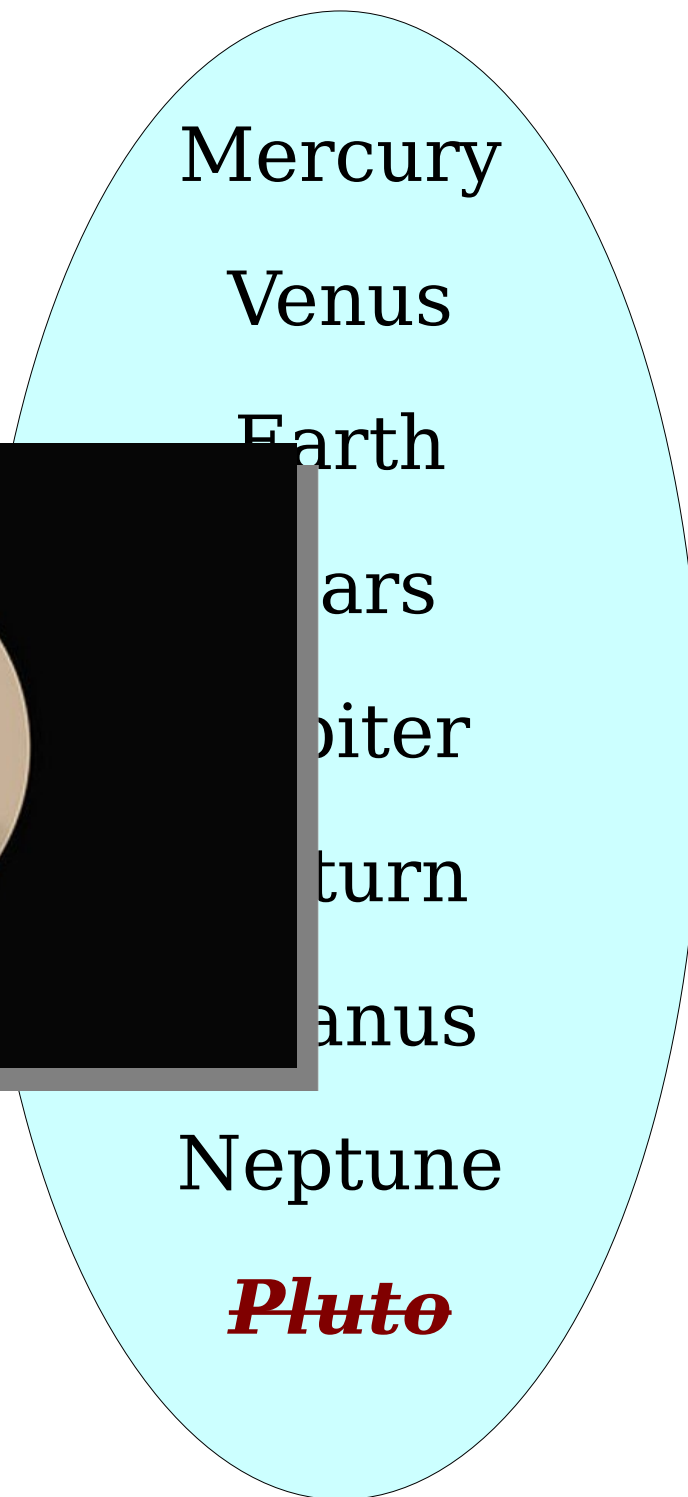
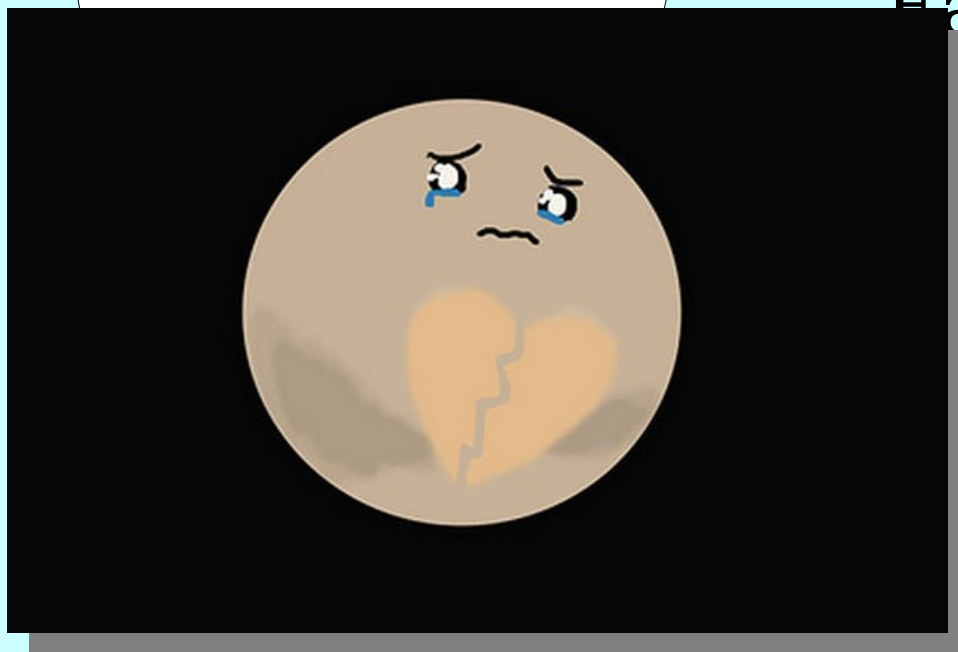
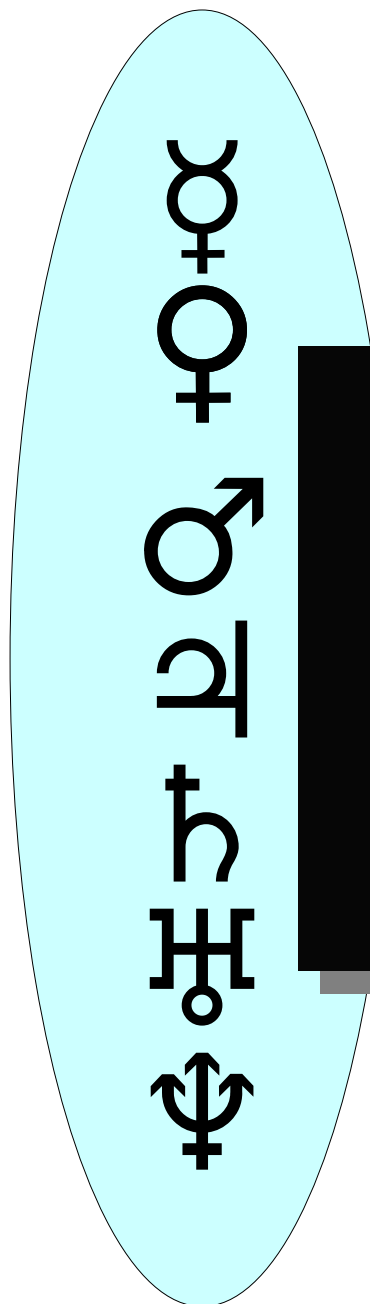
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Mercury

Venus

Earth

Mars

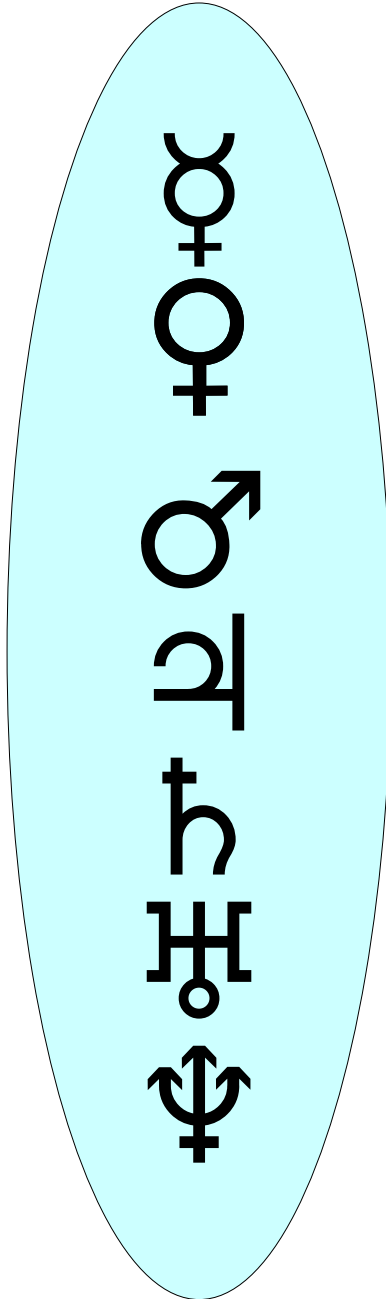
Jupiter

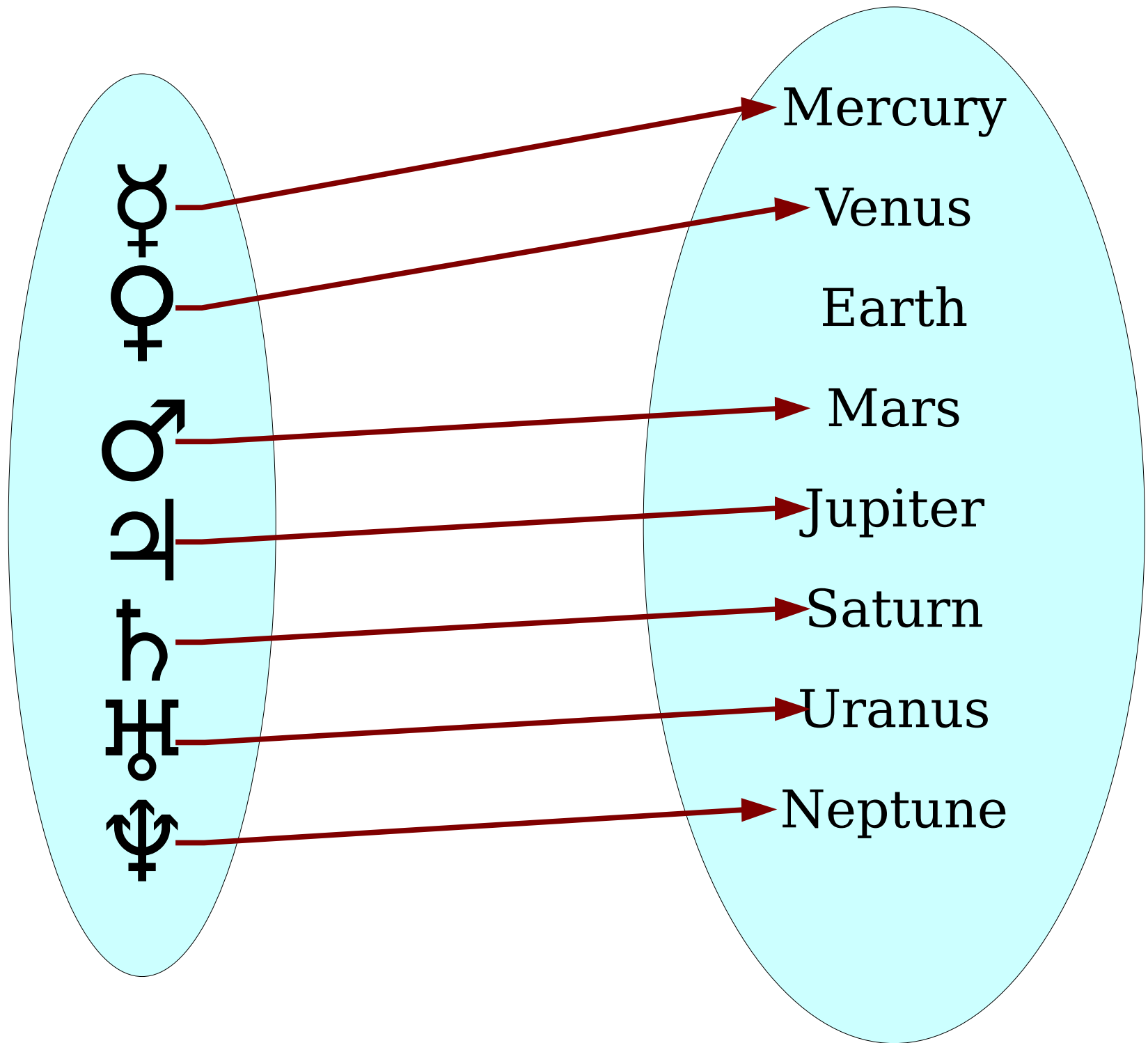
Saturn

Uranus

Neptune

Pluto





Injective Functions

- A function $f : A \rightarrow B$ is called **injective** (or **one-to-one**) when the following statement is true about f :

$$\forall a_1 \in A. \forall a_2 \in A. (a_1 \neq a_2 \rightarrow f(a_1) \neq f(a_2))$$

(“If the inputs are different, the outputs are different.”)

- The following first-order definition is equivalent (*why?*) and is often useful in proofs.

$$\forall a_1 \in A. \forall a_2 \in A. (f(a_1) = f(a_2) \rightarrow a_1 = a_2)$$

(“If the outputs are the same, the inputs are the same.”)

- A function with this property is called an **injection**.
- How does this compare to our second rule for functions?

Inject

Second, f must be deterministic:

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Injections

- Let S be the set of all CS103 students. Which of the following are injective?
 - $f: S \rightarrow \mathbb{N}$ where $f(x)$ is x 's Stanford ID number.
 - $g: S \rightarrow C$, where C is the set of all continents and $g(x)$ is x 's continent of birth.
 - $h: S \rightarrow N$, where N is the set of all given (first) names, where $h(x)$ is x 's given (first) name.

Answer at

cs103.stanford.edu/poll

$f: A \rightarrow B$ is **injective** when either equivalent statement is true:

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Proofs on Injections

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Theorem: Let $f : \mathbb{N} \rightarrow \mathbb{N}$ be defined as $f(n) = 2n + 7$.
Then f is injective.

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Good exercise: Repeat this proof using the other definition of injectivity!

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$A \leftrightarrow B$		Prove $A \rightarrow B$ and $B \rightarrow A$.
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Functions

Part 1

1. Announcements
2. Functions: Intuitions and Examples
3. Domains and Codomains
4. Official Rules for Functions
5. Ways to Define Functions
6. Involutions (with Proofs!)
7. The Assume-Prove Table
- 8. Injections, Surjections, and Bijections**
9. What's Next?



Still Here

Lassen Peak

Mt. Shasta

Crater Lake

Mt. McLoughlin

Mt. Hood

Mt. St. Helens

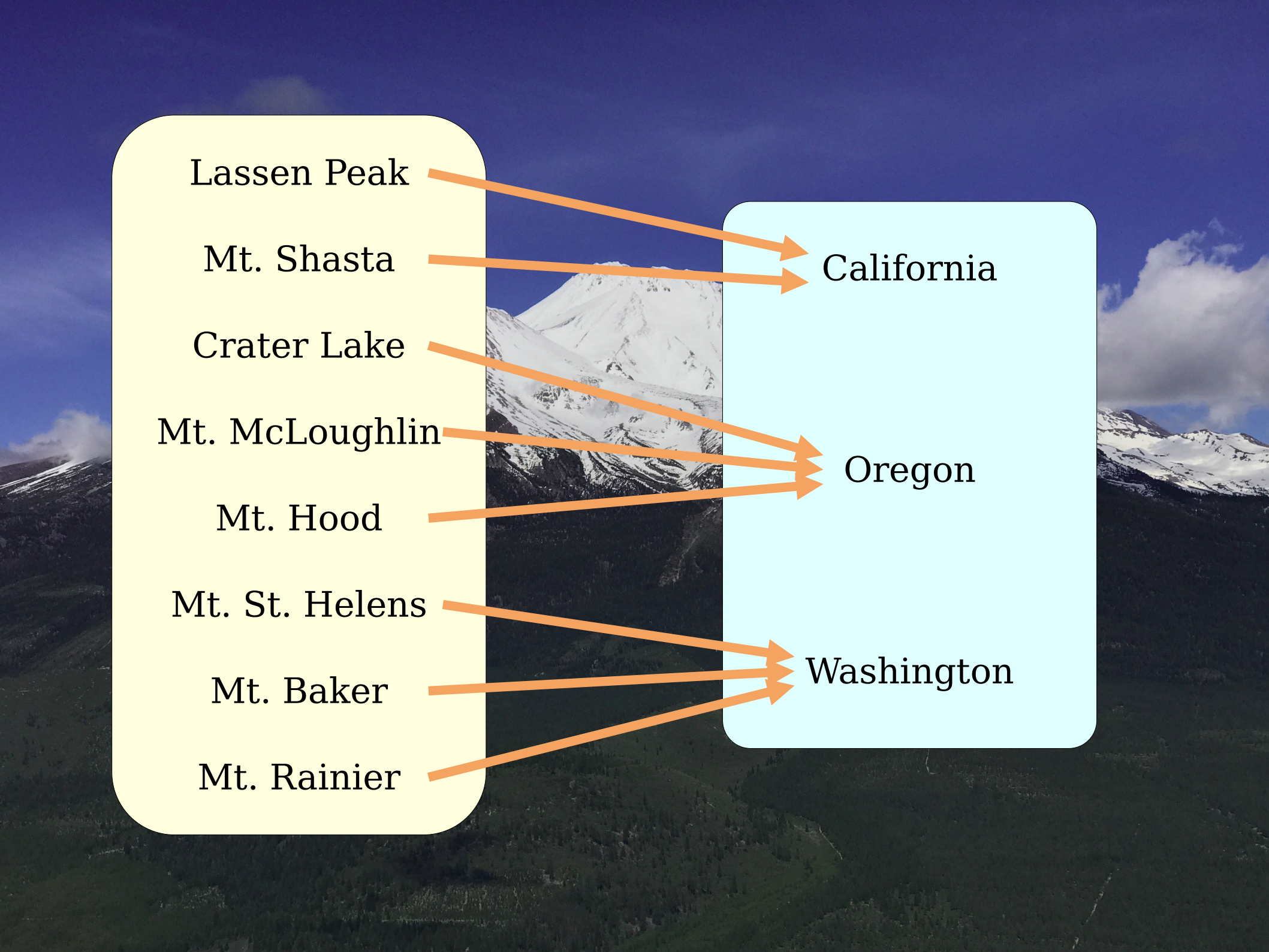
Mt. Baker

Mt. Rainier

California

Oregon

Washington



Surjective Functions

- A function $f : A \rightarrow B$ is called **surjective** (or **onto**) when this first-order logic statement is true about f :

$$\forall b \in B. \exists a \in A. f(a) = b$$

*(“For every possible output,
there's an input that produces it.”)*

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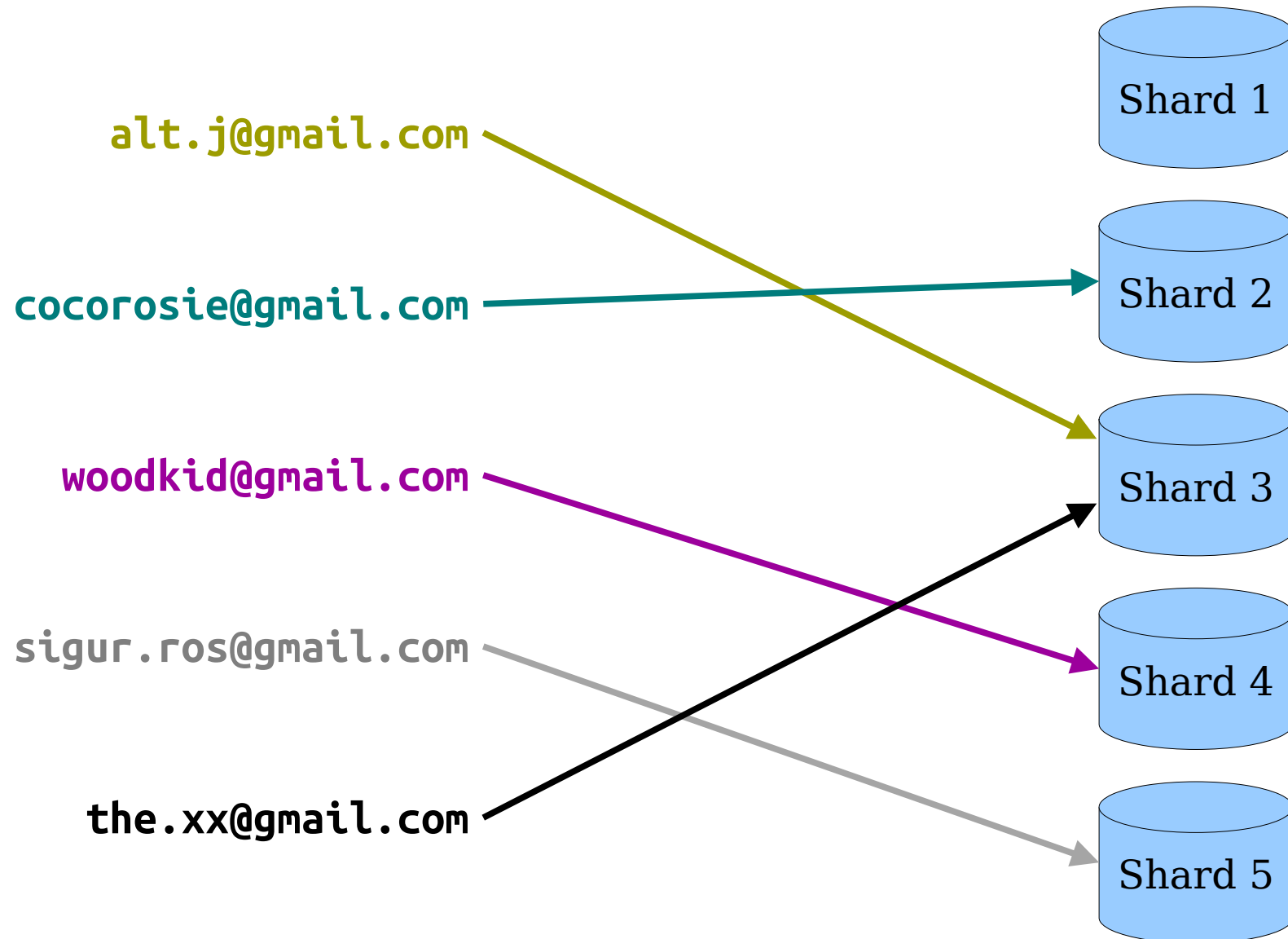
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First, f must obey its domain/codomain rules:

$$\forall a \in A. \exists b \in B. f(a) = b$$

(“Every input in A maps to some output in B .”)

Example: Database Sharding





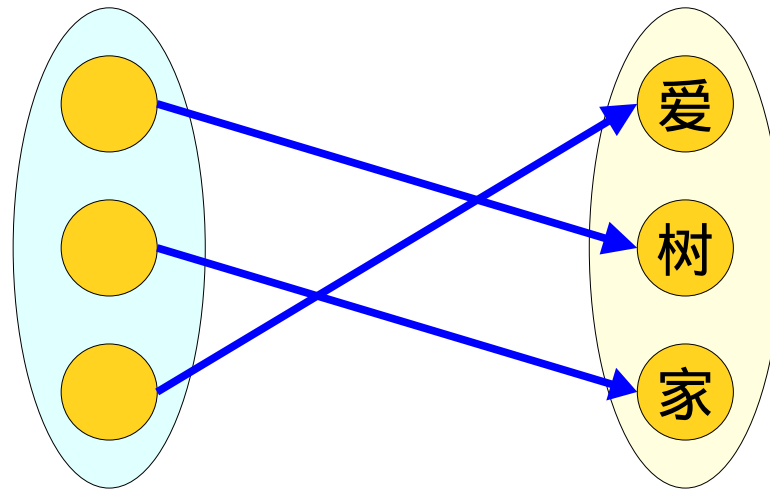
***See appendix for sample
proofs involving
surjections!***

Injective and Surjective

- An injective function associates *at most* one element of the domain with each element of the codomain.
- A surjective function associates *at least* one element of the domain with each element of the codomain.
- What about functions that associate *exactly one* element of the domain with each element of the codomain?

Bijections

- A **bijection** is a function that is both injective and surjective.
- Intuitively, if $f : A \rightarrow B$ is a bijection, then f represents a way of pairing off elements of A and elements of B .



Bijections

- Which of the following are bijections?
 - $f : \mathbb{R} \rightarrow \mathbb{R}$ defined as $f(x) = x$.
 - $f : \mathbb{Z} \rightarrow \mathbb{R}$ defined as $f(x) = x$.
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 - $f : \mathbb{Z} \rightarrow \mathbb{Z}$ defined as $f(x) = 2x + 1$. *Nope!*

A **bijection** is a function that is both injective and surjective.

Bijections

- Which of the following are bijections?
 - $f : \mathbb{R} \rightarrow \mathbb{R}$ defined as $f(x) = x$. *Yep!*
 - $f : \mathbb{Z} \rightarrow \mathbb{R}$ defined as $f(x) = x$. *Nope!*
 - $f : \mathbb{R} \rightarrow \mathbb{R}$ defined as $f(x) = 2x + 1$. *Yep!*
 - $f : \mathbb{Z} \rightarrow \mathbb{Z}$ defined as $f(x) = 2x + 1$. *Nope!*

A ***bijection*** is a function that is both injective and surjective.

Next Time

- ***First-Order Assumptions***
 - The difference between assuming something is true and proving something is true.
- ***Connecting Function Types***
 - Involutions, injections, and surjections are related to one another. How?
- ***Function Composition***
 - Sequencing functions together.

Appendix:
More Proofs on Functions

Proof 1: Proving a function is surjective.

Surjective Functions

Theorem: Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be defined as $f(x) = 2x$. Then $f(x)$ is surjective.

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What does it mean for f to be surjective?

$$\forall y \in \mathbb{R}. \exists x \in \mathbb{R}. f(x) = y$$

Therefore, we'll choose an arbitrary $y \in \mathbb{R}$, then prove that there is some $x \in \mathbb{R}$ where $f(x) = y$.

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Theorem: Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be defined as $f(x) = 2x$. Then $f(x)$ is surjective.

Proof: Consider any $y \in \mathbb{R}$. We will prove that there is a choice of $x \in \mathbb{R}$ such that $f(x) = y$.

Let $x = y / 2$.

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Let $x = y / 2$. Then we see that

$$f(x) = f(y / 2)$$

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This proof contains no first-order logic syntax (quantifiers, connectives, etc.). It's written in plain English, just as usual.

Proof 2: Proving a function is not surjective.

Surjective Functions

Theorem: Let $g : \mathbb{N} \rightarrow \mathbb{N}$ be defined as $g(n) = 2n$. Then $g(x)$ is not surjective.

Surjective Functions

Theorem: Let $g : \mathbb{N} \rightarrow \mathbb{N}$ be defined as $g(n) = 2n$. Then $g(x)$ is not surjective.

What does it mean for g to be surjective?

$$\forall n \in \mathbb{N}. \exists m \in \mathbb{N}. g(m) = n$$

What is the negation of the above statement?

$$\neg \forall n \in \mathbb{N}. \exists m \in \mathbb{N}. g(m) = n$$

$$\exists n \in \mathbb{N}. \neg \exists m \in \mathbb{N}. g(m) = n$$

$$\exists n \in \mathbb{N}. \forall m \in \mathbb{N}. g(m) \neq n$$

Therefore, we need to find a natural number n where, regardless of which m we pick, we have $g(m) \neq n$.

Surjective Functions

Theorem: Let $g : \mathbb{N} \rightarrow \mathbb{N}$ be defined as $g(n) = 2n$. Then $g(x)$ is not surjective.

Proof:

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Proof: Let $n = 137$.

Surjective Functions

Theorem: Let $g : \mathbb{N} \rightarrow \mathbb{N}$ be defined as $g(n) = 2n$. Then $g(x)$ is not surjective.

Proof: Let $n = 137$.

Our overall goal is to prove

$$\exists n \in \mathbb{N}. \forall m \in \mathbb{N}. g(m) \neq n.$$

We just made our choice of n .

Therefore, we need to prove

$$\forall m \in \mathbb{N}. g(m) \neq n.$$

We'll therefore pick an arbitrary $m \in \mathbb{N}$, then prove that $g(m) \neq n$.

Surjective Functions

Theorem: Let $g : \mathbb{N} \rightarrow \mathbb{N}$ be defined as $g(n) = 2n$. Then $g(x)$ is not surjective.

Proof: Let $n = 137$. Now, pick an arbitrary $m \in \mathbb{N}$.

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Notice that $g(m) = 2m$ is even, while 137 is odd.

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